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Reports of the Department of Geodetic Science

Report No. 92

KINEMATICAL GEODESY

by

Helmut Moritz

Prepared for

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts 01730

Contract No. AF19(628)-5701
Project No. 7600
Task No. 760002, 04
Work Unit No. 76000201, 76000401

Scientific Report No. 16

Contract Monitor: Bela Szabo
Terrestrial Sciences Laboratory

The Ohio State University
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73

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FOREWORD

This report was prepared by Helmut Moritz, Professor, Technische Universität Berlin, and Research Associate, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. AF19(628)-5701, OSURF Project No. 2122, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen W. Williams and Mr. Bela Szabo, Project Scientists.

ABSTRACT

With the use of moving instruments, such as airborne gravimeters, which are often related to precise inertial stabilization systems, the essentially statical character of geodesy begins to be enlarged by kinematical features. The potential of gravity, combining gravitational attraction and centrifugal force, is no longer adequate for kinematics, but other inertial forces of Coriolis type must also be considered.

The main purpose of the present paper is an investigation of the geodetic aspects of the interrelation of gravitational and inertial forces and their separation by means of structural differences in their respective fields. For a deeper insight, the general theory of relativity is indispensable; it also furnishes convenient mathematical techniques.

The extraction of purely gravitational effects is possible with second and third derivatives of the potential; therefore geodetic applications of these quantities are discussed, integral formulas similar to Stokes' integral being given for this purpose.

TABLE OF CONTENTS

KINEMATICAL GEODESY

Introduction

Until recently, geodetic measurements were usually performed with stationary instruments, that is, instruments at rest on the earth's surface, such as theodolites or gravimeters. Therefore, only the statical features of the earth and its gravity field were of geodetic interest. A typical example is the omission of the Coriolis force from the gravity field of the earth, because this force acts on moving bodies only.

With the advent of artificial satellites and precise inertial navigation systems, and with the development of shipborne and airborne systems for gravity measurement, the element of time is now entering into physical geodesy; the essentially statical character of geodesy begins to be enlarged by kinematical features. As a simple example, Coriolis forces show up in marine and aerial gravimetry. This indicates that the potential of gravity, combining gravitational attraction and centrifugal force but excluding the Coriolis force, is no longer adequate for treating certain kinematical effects of the earth's gravitational field.

The Principle of Equivalence, as expressed by the identity of inertial and gravitational mass, makes it possible to combine gravitational attraction and centrifugal force, which is an inertial effect, into a unified force, gravity. The same principle, however, also makes it extremely difficult to separate the effect of undesirable inertial forces from the genuine gravitational forces, in which we are interested in gravity measurement.

This problem of separation of inertial and gravitational forces is important if the measuring system is exposed to irregular accelerations, such as aboard a ship or an aircraft. It does not occur in considering the motion of a satellite. Since the main purpose of the present report is an investigation of the geodetic aspects of the interrelation of gravitation and inertia, satellite dynamics is naturally excluded here.

In a sense, the problems of aerial gravimetry and inertial positioning (navigation) are inverse to each other. In aerial gravimetry, outside geodetic information is used to separate gravitational from inertial effects in order to measure gravity (or other quantities of the gravitational field). In inertial navigation, the gravitational field is needed as an input to obtain position (which, applied to geodesy, means geodetic position). Schematically we may write

$$\left.\begin{array}{l}\text{Aerial Gravimetry}\\ \text{Geodetic Position}\end{array}\right\} \Longrightarrow \quad \text{Gravity Field;}$$

$$\left.\begin{array}{l}\text{Inertial Navigation}\\ \text{Gravity Field}\end{array}\right\} \Longrightarrow \quad \text{Geodetic Position.}$$

Separating gravitational from inertial effects wholly without outside (noninertial) information is impossible; it would be a bootstrap operation to get the positional information in aerial gravimetry wholly from inertial navigation. Still, the degree to which such outside information is needed varies considerably from method to method. Auxiliary factors that permit a certain discrimination between gravitational and inertial effects are:

A. Statistical frequency behaviour. The disturbing accelerations usually have a higher frequency than gravity; thus they may be removed, to a certain extent, by filtering techniques.

B. Structural differences between the gravitational and the inertial fields. The second field is more regular than the first.

Filtering techniques based on statistical behaviour are discussed in another report (Moritz, 1967). Here we shall investigate the mathematical structure of the fields involved and geodetic applications.

It turns out that factor B has no effect on the gravity g (or more generally speaking, on the first derivatives of the potential), so that factor A must be used with measurements of g. It is only in second and higher derivatives that the structural differences between gravitational and inertial fields will contribute to their separation.

For this reason, we shall in Chapter 1 give a theoretical (perhaps somewhat unconventional) outline of possible geodetic uses of second and third derivatives of the potential.

Chapter 2 is devoted to theoretical aspects of measurement of these derivatives, with emphasis on separability of inertial and gravitational effects because of the structure of their respective fields. The approach through classical mechanics is simple and straightforward. However, since the Principle of Equivalence cannot be explained on the basis of classical mechanics, but can be understood only on the basis of Einstein's general theory of relativity, there will always be the shadow of a doubt in arguments involving classical mechanics only; we shall try

to remove this shadow by a rather detailed study of the relativistic aspects of the problem.

Time is obviously the key concept in the kinematical[1] applications of geodesy. Since time is a natural part of the space-time of the theory of relativity, we may expect conceptual and mathematical advantages in applying this theory to kinematical geodesy. As an example, certain relativistic quantities generalize the potential so as to contain the Coriolis force as well as gravitational attraction and centrifugal force; see sec. 2.3.1.

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[1] We prefer the term "kinematical" to "dynamical" for two reasons. First, "dynamical geodesy" is often used equivalently with "physical geodesy," and second, the general theory of relativity is kinematical rather than dynamical in character.

1. Geodetic Use of Gravitational Gradients

1.1. Second Derivatives

Let us assume that all second derivatives of the gravitational potential V in an earth-fixed coordinate system xyz have been measured along a line which, for instance, represents the flight path. These second derivatives form a matrix

$$\begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix} , \tag{1}$$

where, as usual,

$$V_{xx} = \frac{\partial^2 V}{\partial x^2} , \quad V_{xy} = \frac{\partial^2 V}{\partial x \partial y} , \quad \text{etc.} \tag{2}$$

This matrix is symmetric because

$$V_{yx} = V_{xy} , \quad \text{etc.}$$

If all second derivatives have been measured, so that the whole matrix (1) is known, then the first derivatives, the components of the vector of gravitational force,

$$\vec{F} = (V_x, V_y, V_z) \equiv \text{grad } V, \tag{3}$$

are obtained by integration along the flight path. For instance,

$$V_x = (V_x)_0 + \int_{P_0}^{P} (V_{xx}dx + V_{xy}dy + V_{xz}dz) .$$

$(V_x)_0$ refers to an initial point P_0 on the flight path, and V_x refers to a current point P. It is understood that the flight path is known as a function of time t:

$$\begin{aligned} x &= x(t) , \\ y &= y(t) , \\ z &= z(t) . \end{aligned} \tag{4}$$

Then

$$dx = \dot{x}\,dt \text{ , etc.}$$

(the dot denoting differentiation with respect to time), so that we obtain

$$
\begin{aligned}
V_x &= (V_x)_o + \int_{P_o}^{P} (V_{xx}\dot{x} + V_{xy}\dot{y} + V_{xz}\dot{z})\,dt, \\
V_y &= (V_y)_o + \int_{P_o}^{P} (V_{yx}\dot{x} + V_{yy}\dot{y} + V_{yz}\dot{z})\,dt, \\
V_z &= (V_z)_o + \int_{P_o}^{P} (V_{zx}\dot{x} + V_{zy}\dot{y} + V_{zz}\dot{z})\,dt .
\end{aligned}
\tag{5}
$$

Another integration gives the potential V itself:

$$V = V_o + \int_{P_o}^{P} (V_x\dot{x} + V_y\dot{y} + V_z\dot{z})\,dt \quad . \tag{6}$$

The initial data $(V_x)_o$, $(V_y)_o$, $(V_z)_o$, V_o must be assumed to be known.

It is characteristic for this method that the data (1) are needed <u>only along a line</u> (the flight path). External geodetic information, as mentioned in the Introduction, is required in this method to obtain the flight path (4), which is necessary to perform the integrations (5) and (6).

1.2. <u>Second Horizontal Derivatives</u>

We consider the quantities

$$W_{yy} - W_{xx} \equiv W_\Delta \text{ and } W_{xy} \quad , \tag{7}$$

which can be measured with a torsion balance (Baeschlin, 1948; Mueller, 1963). Here W denotes the gravity potential, the sum of the gravitational potential V and the potential of centrifugal force. The derivatives are taken with respect to a local coordinate system having a vertical z-axis, so that the quantities entering in (7) are horizontal derivatives.

As usual, we define the anomalous potential T as the difference of W and the normal gravity potential U:

$$T = W - U. \tag{8}$$

Since the normal values of (7), U_Δ and U_{xy}, are known (Mueller, 1963, p. 147), we obtain from (7) also

$$T_{yy} - T_{xx} = T_\Delta \text{ and } T_{xy} . \tag{9}$$

It is well known that

$$\zeta = \frac{1}{G} T \tag{10}$$

(G representing a mean value of gravity) gives the height anomaly (the separation between corresponding geopotential and spheropotential surfaces), and that

$$\xi = -\frac{1}{G} T_x \ , \ \eta = -\frac{1}{G} T_y \tag{11}$$

give the components of the deflection of the vertical (the x-axis pointing northward).

By (9) and (11), the quantities

$$\begin{aligned} \xi_y &= \eta_x = -\frac{1}{G} T_{xy} \ , \\ \xi_x - \eta_y &= \frac{1}{G} T_\Delta \end{aligned} \tag{12}$$

are obtained from torsion balance measurements. The problem is to find the deflection components ξ and η and the height anomaly ζ by a suitable integration of (12).

This integration is usually performed by a numerical method, the mathematical significance of which is not immediately evident. Mathematically, the integration of (12) poses an interesting and not too difficult problem in partial

differential equations, which we shall consider now. Apart from the theoretical interest, the mathematical insight gained in this way may also be of indirect practical use.

We shall limit our considerations to a restricted part of the earth's surface, which may for the present purpose be assumed to be plane. The quantities (12) are assumed to be given at each point of this plane area, as functions of x and y:

$$\begin{aligned} \xi_y &= \eta_x = \varphi(x,y) \;, \\ \xi_x - \eta_y &= \psi(x,y) \;, \end{aligned} \tag{13}$$

where we have put

$$\varphi(x,y) = -\frac{1}{G} T_{xy} \;, \quad \psi(x,y) = \frac{1}{G} T_{\Delta} \;. \tag{14}$$

The problem is to solve the system of partial differential equations (13) for ξ and η, that is, to find functions

$$\xi = \xi(x,y) \;, \quad \eta = \eta(x,y)$$

satisfying (13).

Conceptually it would even be simpler to consider the system of partial differential equations

$$\begin{aligned} T_{xy} &= -G\varphi(x,y) \;, \\ T_{xx} - T_{yy} &= -G\psi(x,y) \;, \end{aligned} \tag{15}$$

equivalent to (13) but containing only one unknown function T, but the solution of (13) is easier.

The general solution of the first equation of (13) must have the form

$$\xi = \int_{y_o}^{y} \varphi(x, y')\, dy' + F(x) \quad , \tag{16}$$

where $F(x)$ is an arbitrary function of x. This is readily verified by substitution. For reasons of convenience we put

$$F(x) = A + B(x-x_o) + U(x) \quad ,$$

where A and B are arbitrary constants and $U(x)$ satisfies the conditions

$$U(x_o) = U_x(x_o) = 0 \tag{17a}$$

but is arbitrary otherwise.

Thus we obtain from (16), adjoining an analogous equation for η,

$$\left.\begin{aligned} \xi &= \int_{y_o}^{y} \varphi(x, y')\, dy' + A + B(x-x_o) + U(x) \quad , \\ \eta &= \int_{x_o}^{x} \varphi(x', y)\, dx' + C + D(y-y_o) + V(y) \quad , \end{aligned}\right\} \tag{18}$$

where $V(y)$ satisfies

$$V(y_o) = V_y(y_o) = 0 \tag{17b}$$

but is arbitrary otherwise.

Now the functions $U(x)$ and $V(y)$ are determined by means of the second equation of (13). The insertion of (18) gives

$$\begin{aligned} \xi_x - \eta_y &= \int_{y_o}^{y} \varphi_x(x, y')\, dy' - \int_{x_o}^{x} \varphi_y(x', y)\, dx' + \\ &\quad + B - D + U_x(x) - V_y(y) = \psi(x, y) \quad . \end{aligned} \tag{19}$$

Setting $y = y_o$ in this equation and considering (17b) we obtain

$$U_x(x) = \int_{x_o}^{x} \varphi_y(x', y_o)\, dx' + \psi(x, y_o) - B + D \quad .$$

Since $U_x = dU/dx$, this equation may be integrated to give

$$U(x) = \int_{x_0}^{x} (x-x')\,\varphi_y(x', y_0)dx' + \int_{x_0}^{x} \psi(x', y_0)\,dx' + (D-B)\,(x-x_0) \quad .$$

Here we have used the fact that

$$\frac{d}{dx}\int_{a}^{x} (x-x')\,f(x')\,dx' = \int_{a}^{x} f(x')\,dx' \tag{20}$$

and considered (17a). In exactly the same way (by setting $x = x_0$ in (19) and integrating) we find $V(y)$, so that (18) becomes

$$\begin{aligned} \xi &= \int_{y_0}^{y} \varphi(x, y')\,dy' + \int_{x_0}^{x} (x-x')\,\varphi_y\,(x', y_0)\,dx' + \\ &\quad + \int_{x_0}^{x} \psi(x' y_0)\,dx' + A + D(x-x_0) \quad , \\ \eta &= \int_{x_0}^{x} \varphi(x', y)\,dx' + \int_{y_0}^{y} (y-y')\,\varphi_x(x_0, y')\,dy' - \\ &\quad - \int_{y_0}^{y} \psi(x_0, y')\,dy' + C + B(y-y_0) \quad . \end{aligned} \tag{21}$$

To further investigate this solution, we form the derivatives

$$\begin{aligned} \xi_x &= \int_{y_0}^{y} \varphi_x(x, y')\,dy' + \int_{x_0}^{x} \varphi_y(x', y_0)\,dx' + \psi\,(x, y_0) + D \quad , \\ \eta_y &= \int_{x_0}^{x} \varphi_y(x', y)\,dx' + \int_{y_0}^{y} \varphi_x(x_0, y')\,dy' - \psi(x_0, y) + B \quad . \end{aligned} \tag{22}$$

Now

$$\int_{y_0}^{y} \varphi_x(x, y')\,dy' - \int_{y_0}^{y} \varphi_x(x_0, y')\,dy' = \int_{x_0}^{x}\int_{y_0}^{y} \varphi_{xx}\,(x', y')\,dx'dy'$$

and

$$B - D = \psi(x_0, y_0) \quad , \tag{23}$$

as may be seen by setting $x = x_0$, $y = y_0$ in (19). Thus we find from (22)

$$\xi_x - \eta_y = \int_{x_0}^{x}\int_{y_0}^{y} [\varphi_{xx}(x',y') - \varphi_{yy}(x',y')]\, dx'dy' +$$

$$+ \psi(x,y_0) + \psi(x_0,y) - \psi(x_0,y_0) \ .$$

With the identity

$$\psi(x,y_0) + \psi(x_0,y) - \psi(x_0,y_0) = \psi(x,y) - \int_{x_0}^{x}\int_{y_0}^{y} \psi_{xy}(x',y')\, dx'dy'$$

this becomes

$$\xi_x - \eta_y = \psi(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} [\varphi_{xx}(x',y') - \varphi_{yy}(x',y') - \psi_{xy}(x',y')]\, dx'dy' \ .$$

This agrees with the second equation of (13) only if

$$\varphi_{xx} - \varphi_{yy} - \psi_{xy} \equiv 0 \ . \qquad (24)$$

The equation (24) thus constitutes the <u>integrability condition</u> of the system (13). The data functions $\varphi(x,y)$ and $\psi(x,y)$ cannot be prescribed arbitrarily but must satisfy the integrability condition (24). Errorless data will automatically satisfy (24) because by (14)

$$\varphi_{xx} - \varphi_{yy} - \psi_{xy} = \frac{1}{G}(-T_{xyxx} + T_{xyyy} - T_{yyxy} + T_{xxxy})$$

$$= \frac{1}{G}(-T_{xxxy} + T_{xyyy} - T_{xyyy} + T_{xxxy}) \equiv 0 \ .$$

With empirical data, (24) must be enforced by an adjustment if necessary.

To determine the constants A, B, C, D in (21), we put $x = x_0$, $y = y_0$ in (21) and (22). Using (23) we readily obtain with $\xi_0 = \xi(x_0, y_0)$, etc.:

$$A = \xi_0 , \quad B = (\xi_x)_0 , \quad C = \eta_0 , \quad D = (\eta_y)_0 \ .$$

Only three of these constants are independent, because B and D are connected

by (23), so that

$$D = (\eta_y)_0 = (\xi_x)_0 - \psi(x_0, y_0) .$$

Thus (21) assumes the final form

$$\begin{aligned}
\xi &= \xi_0 + [(\xi_x)_0 - \psi(x_0, y_0)] (x-x_0) + \\
&\quad + \int_{y_0}^{y} \varphi(x, y') \, dy' + \int_{x_0}^{x} (x-x') \, \varphi_y (x', y_0) \, dx' + \int_{x_0}^{x} \psi(x', y_0) \, dx' , \\
\eta &= \eta_0 + (\xi_x)_0 (y-y_0) + \\
&\quad + \int_{x_0}^{x} \varphi(x', y) \, dx' + \int_{y_0}^{y} (y-y') \, \varphi_x (x_0, y') \, dy' + \int_{y_0}^{y} \psi (x_0, y') \, dy' .
\end{aligned} \tag{25}$$

The three quantities ξ_0, η_0, $(\xi_x)_0$ must be given at the initial point P_0, and the data functions φ and ψ must satisfy the integrability condition (24). Then ξ and η are given by (25). If we wish, we may use ξ and η so obtained to get ζ or T by the usual method of astronomical leveling:

$$\zeta = \zeta_0 - \int_{P_0}^{P} (\xi dx + \eta dy) . \tag{26}$$

It should be noted that the integrations needed to determine ξ by (25) are extended over the horizontal or the vertical part of the full line in Fig. 1, whereas the integrations for η involve the broken line in Fig. 1.

This method presupposes the functions $\varphi(x, y)$ and $\psi(x, y)$ to be given at every point of a certain region, thus sharing a general characteristic of physical geodesy. It is theoretically rigorous, apart from the approximation of the reference ellipsoid in a small region by a plane. Extensions to a spherical or ellipsoidal surface of reference, although simple in principle, lead to awkward formulas.

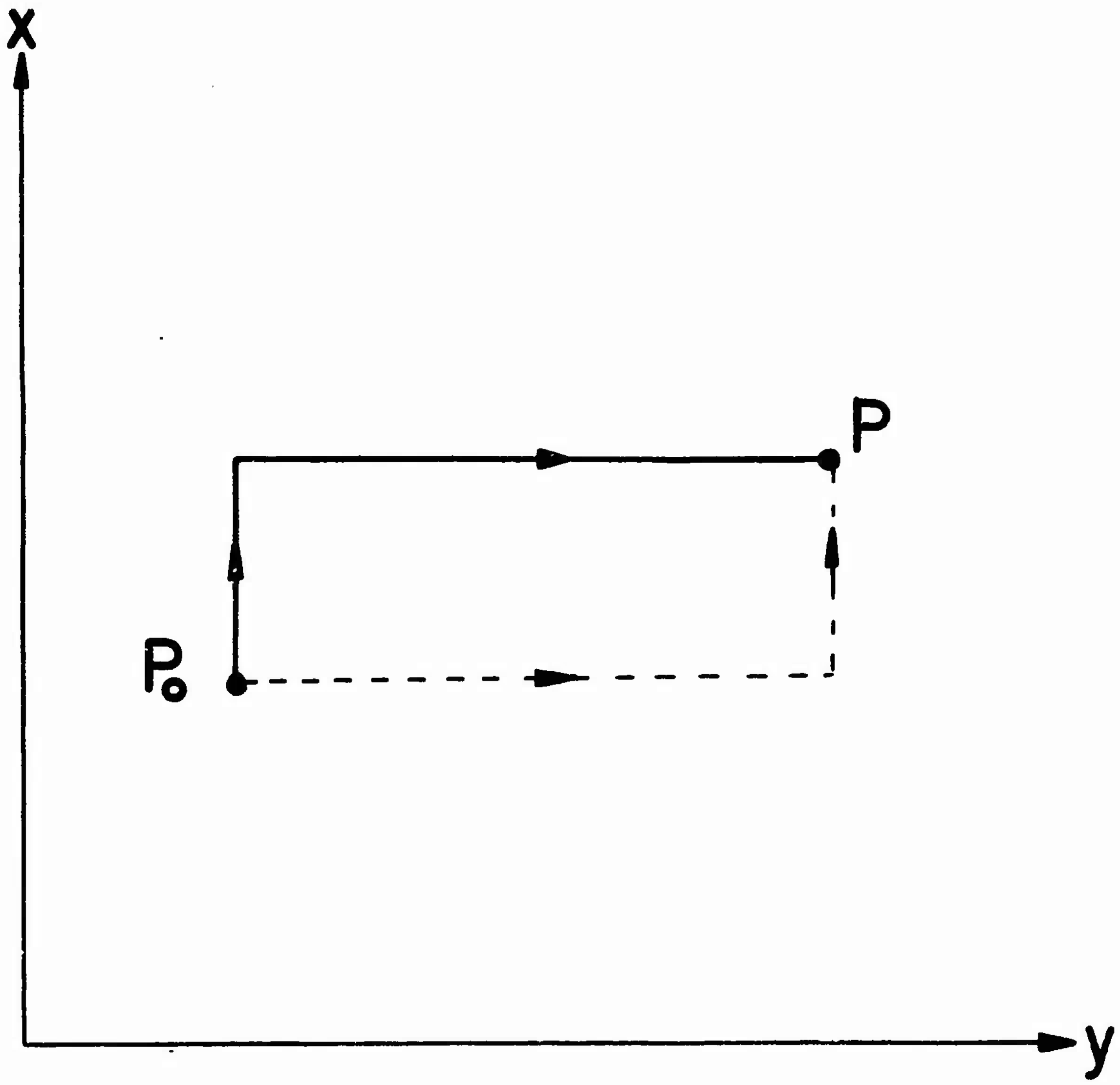

Figure 1

As a matter of fact, measurements of φ and ψ are performed at discrete points only, so that the integrals in (25) must be replaced by sums. It seems preferable, however, to start from rigorous expressions such as (25) and find practical approximations for them, rather than starting from approximations right away.

It is clear that instead of prescribing ξ_o, η_o, $(\xi_x)_o$ it is better in practice to prescribe ξ_o, η_o and, say ξ_1 at a different point P_1. The expression of ξ_o, η_o, $(\xi_x)_o$ by ξ_o, η_o, ξ_1 or by ξ_o, η_o, η_1 by means of (25) is straightforward and may be left to the reader.

Finally we remark that the horizontal derivatives (9) may also be measured by airborne equipment. If these quantities are measured in this way along a level surface, formulas (25) and (26) may be applied to get T; by adding the normal gravitational potential to T, the potential V is obtained.

1.3. Second Vertical Derivatives

Using the notations of the preceding section, we consider the quantity

$$T_{zz} \ .$$

Since the z-axis coincides with the local vertical, this is the second vertical derivative of the anomalous potential (8).

Assuming T_{zz} to be known at every point of a level surface, we shall now derive an integral formula expressing T in terms of T_{zz}. We consider the spherical approximation, that is, we replace the reference ellipsoid formally by a sphere; this approximation is also used in Stokes' formula.

Then we have

$$T_{zz} \equiv \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial r^2} = T_{rr} \ , \tag{27}$$

since the normal to a sphere is directed along the radius vector r. We shall now follow a procedure described in (Heiskanen and Moritz, 1967, pp. 88 and 97), using the same notations.

We expand the anomalous potential T into a series of spherical harmonics:

$$T(r,\theta,\lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n(\theta,\lambda) \ . \tag{28}$$

Differentiating twice with respect to r we obtain in space

$$T_{rr}(r,\theta,\lambda) = \sum_{0}^{\infty} (n+1)\ (n+2)\ \frac{R^{n+1}}{r^{n+3}}\ T_n\ (\theta,\lambda) \ ,$$

and at the level surface under consideration (r=R)

$$T_{rr} = \frac{1}{R^2} \sum_{0}^{\infty} (n+1)\ (n+2)\ T_n\ (\theta,\lambda) \ . \tag{29}$$

We may also directly express T_{rr} as a series of Laplace's surface harmonics:

$$T_{rr} = \sum_{0}^{\infty} T_{rr,n}(\theta,\lambda) \ .$$

Comparing these two series yields

$$T_{rr,n}(\theta,\lambda) = \frac{(n+1)\ (n+2)}{R^2}\ T_n\ (\theta,\lambda) \ , \qquad T_n = \frac{R^2}{(n+1)\ (n+2)}\ T_{rr,n} \ ,$$

so that on the level surface (r=R)

$$T = \sum_{0}^{\infty} T_n\ (\theta,\lambda) = \sum_{0}^{\infty} \frac{R^2}{(n+1)\ (n+2)}\ T_{rr,n} \ . \tag{30}$$

According to a well-known formula for spherical harmonics (Heiskanen and Moritz, 1967, p. 30) we have

$$T_{rr,n} = \frac{2n+1}{4\pi} \iint_\sigma T_{rr} P_n (\cos\psi)\, d\sigma ,$$

so that (30) becomes

$$T = \frac{R^2}{4\pi} \sum_0^\infty \frac{2n+1}{(n+1)(n+2)} \iint_\sigma T_{rr} P_n (\cos\psi)\, d\sigma .$$

By interchanging the order of summation and integration (which may be shown to be permissible) we get

$$T = \frac{R^2}{4\pi} \iint_\sigma \left[\sum_0^\infty \frac{2n+1}{(n+1)(n+2)} P_n (\cos\psi) \right] T_{rr}\, d\sigma .$$

On putting

$$S_1(\psi) = \sum_0^\infty \frac{2n+1}{(n+1)(n+2)} P_n (\cos\psi) \tag{31}$$

we have

$$T = \frac{R^2}{4\pi} \iint_\sigma T_{rr} S_1 (\psi)\, d\sigma . \tag{32}$$

This integral formula is completely analogous to Stokes' formula: it expresses T in terms of T_{rr} just as Stokes' formula expresses T in terms of Δg. Even the expression (31) for the function $S_1(\psi)$ is comparable to the expression of Stokes' function $S(\psi)$ in terms of spherical harmonics (Heiskanen and Moritz, 1967, p. 97).

We shall now develop a closed expression for the function $S_1(\psi)$ by summing the series (31). The function

$$f(t,\psi) = \frac{1}{\ell} = \frac{1}{\sqrt{1-2t\cos\psi+t^2}} \tag{33}$$

may be expanded as

$$f(t,\psi) = \sum_0^\infty t^n P_n(\cos\psi) \tag{34}$$

(Heiskanen and Moritz, 1967, p. 33). By consecutive integrations with respect to t we obtain

$$\begin{aligned}
f_1(x,\psi) &= \int_0^x f(t,\psi)\,dt = \sum_0^\infty \frac{x^{n+1}}{n+1} P_n(\cos\psi) \;, \\
f_2(x,\psi) &= \int_0^x f_1(t,\psi)\,dt = \sum_0^\infty \frac{x^{n+2}}{(n+1)\,(n+2)} P_n \;, \\
f_3(x,\psi) &= \int_0^x f_2(t,\psi)\,dt = \sum_0^\infty \frac{x^{n+3}}{(n+1)\,(n+2)\,(n+3)} P_n \;.
\end{aligned} \tag{35}$$

According to (Smirnow, 1966, p. 46) we have

$$\begin{aligned}
f_2(t,\psi) &= \int_0^x (x-t)\, f(t)\,dt \;, \\
f_3(t,\psi) &= \frac{1}{2} \int_0^x (x-t)^2 f(t)\,dt \;.
\end{aligned} \tag{36}$$

From (35) and (36) we obtain with $x = 1$ and $1/\ell$ defined by (33):

$$\begin{aligned}
h_1(\psi) &\equiv \sum_0^\infty \frac{P_n(\cos\psi)}{n+1} = \int_0^1 \frac{dt}{\ell} \;, \\
h_2(\psi) &\equiv \sum_0^\infty \frac{P_n(\cos\psi)}{(n+1)\,(n+2)} = \int_0^1 \frac{dt}{\ell} - \int_0^1 \frac{t}{\ell}\,dt \;, \\
h_3(\psi) &\equiv \sum_0^\infty \frac{P_n(\cos\psi)}{(n+1)\,(n+2)\,(n+3)} = \frac{1}{2} \int_0^1 \frac{dt}{\ell} - \int_0^1 \frac{t}{\ell}\,dt + \frac{1}{2} \int_0^1 \frac{t^2}{\ell}\,dt \;.
\end{aligned} \tag{37}$$

The integrations, to be performed by standard methods, give

$$\begin{aligned}
h_1(\psi) &= \ell n(1+1/\sin\frac{\psi}{2}) \;, \\
h_2(\psi) &= 2\sin^2\frac{\psi}{2}\, \ell n(1+1/\sin\frac{\psi}{2}) + 1 - 2\sin\frac{\psi}{2} \;, \\
h_3(\psi) &= (-\sin^2\frac{\psi}{2} + 3\sin^4\frac{\psi}{2})\, \ell n(1+1/\sin\frac{\psi}{2}) + \\
&\quad + \frac{1}{4} + \frac{3}{2}\sin^2\frac{\psi}{2} - 3\sin^3\frac{\psi}{2} \;.
\end{aligned} \tag{38}$$

After these mathematical preliminaries we can at once determine the sum (31). We have

$$S_1(\psi) = \Sigma \frac{2n+1}{(n+1)(n+2)} P_n(\cos\psi)$$

$$= \Sigma\left(\frac{2}{n+1} - \frac{3}{(n+1)(n+2)}\right) P_n(\cos\psi)$$

$$= 2h_1(\psi) - 3h_2(\psi) ,$$

so that from (38) we obtain our final result

$$S_1(\psi) = (2-6\sin^2\frac{\psi}{2}) \ln(1+1/\sin\frac{\psi}{2}) - 3 + 6\sin\frac{\psi}{2} . \tag{39}$$

This function is to be used in the integral formula (32).

1.4. Third Vertical Derivatives

The measurement of second derivatives requires a gyroscopically stabilized measuring system. Third derivatives may be obtained without gyroscopic stabilization (sec. 2.2). Therefore we shall develop an integral formula corresponding to (32) but involving third vertical derivatives

$$T_{zzz} = T_{rrr} . \tag{40}$$

We form the third derivative of (28) with respect to r and set r = R, obtaining

$$T_{rrr} = -\frac{1}{R^3} \sum_0^\infty (n+1)(n+2)(n+3) T_n(\theta,\lambda) . \tag{41}$$

By exactly the same process by which we derived (32) from (29) we obtain from (41)

$$T = \frac{R^3}{4\pi} \iint_\sigma T_{rrr} S_2(\psi) d\sigma \tag{42}$$

with

$$S_2(\psi) = -\sum_0^\infty \frac{2n+1}{(n+1)(n+2)(n+3)} P_n(\cos\psi) \tag{43}$$

corresponding to (31).

The series (43) may be readily summed. We have

$$S_2(\psi) = \sum_0^\infty \left(-\frac{2}{(n+1)(n+2)} + \frac{5}{(n+1)(n+2)(n+3)}\right) P_n(\cos\psi)$$

$$= -2h_2(\psi) + 5h_3(\psi) ,$$

using the notation of (37). From (38) we obtain at once our final result

$$S_2(\psi) = \left(-9\sin^2\frac{\psi}{2} + 15\sin^4\frac{\psi}{2}\right) \ln\left(1+1/\sin\frac{\psi}{2}\right) - \\ -\frac{3}{4} + 4\sin\frac{\psi}{2} + \frac{15}{2}\sin^2\frac{\psi}{2} - 15\sin^3\frac{\psi}{2} . \tag{44}$$

This function is to be used in the integral formula (42).

1.5. Transformation of Gradients

In conformity with geodetic usage, we shall use the term "(generalized) gradients" for the second and third derivatives of the potential. The problem of transforming these gradients from one rectangular coordinate system to another occurs frequently; for instance, it may be required to transform them from a system fixed to the measuring equipment to a system fixed to the earth.

It is convenient to use the symbols x_1, x_2, x_3 for the coordinates x, y, z. Let the original system be (x_1', x_2', x_3') , and the new system (x_1, x_2, x_3). The transformation equation may be concisely written as

$$x_i' = \sum_{j=1}^{3} a_{ij} x_j , \tag{45}$$

or as the matrix equation

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{45'}$$

Here the element a_{ij} is the cosine of the angle between the old x_i'-axis and the new x_j-axis.

The matrix (a_{ij}) is orthogonal, that is, its elements satisfy the six conditions

$$\sum_k a_{ik} a_{jk} = \delta_{ij} = \begin{cases} 1, & j = i , \\ 0, & j \neq i . \end{cases} \tag{46}$$

Since the nine elements a_{ij} are connected by six conditions, they must be functions of three independent parameters, corresponding to the three degrees of freedom of rotation. For these parameters one may take the three <u>Eulerian angles</u> θ, φ, ψ in terms of which we have (e.g., Goldstein, 1950, sec. 4-4).

$$\begin{aligned}
a_{11} &= \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta, \\
a_{12} &= \sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\theta, \\
a_{13} &= \sin\psi\sin\theta, \\
a_{21} &= - \cos\varphi\sin\psi - \sin\varphi\cos\psi\cos\theta, \\
a_{22} &= - \sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta, \\
a_{23} &= \cos\psi\sin\theta, \\
a_{31} &= \sin\varphi\sin\theta, \qquad a_{32} = - \cos\varphi\sin\theta, \qquad a_{33} = \cos\theta .
\end{aligned} \tag{47}$$

The gradients in the original system will be denoted by a prime, so that

$$T'_{ij} = \frac{\partial^2 T}{\partial x'_i \partial x'_j} , \qquad T'_{ijk} = \frac{\partial^3 T}{\partial x'_i \partial x'_j \partial x'_k} ; \tag{48a}$$

similarly in the new system

$$T_{ij} = \frac{\partial^2 T}{\partial x_i \partial x_j} , \qquad T_{ijk} = \frac{\partial^2 T}{\partial x_i \partial x_j \partial x_k} . \tag{48b}$$

By the usual rules of partial differentiation we have

$$\frac{\partial T}{\partial x_i} = \sum_{p=1}^{3} \frac{\partial T}{\partial x'_p} \frac{\partial x'_p}{\partial x_i} = \sum_p a_{pi} \frac{\partial T}{\partial x'_p}$$

using (45). Further differentiation yields, on introducing the abbreviations (48a, b),

$$T_{ij} = \sum_{p,q=1}^{3} a_{pi} a_{qj} T'_{pq} , \tag{49}$$

$$T_{ijk} = \sum_{p,q,r=1}^{3} a_{pi} a_{qj} a_{rk} T'_{pqr} . \tag{50}$$

As an example, in sec. 1.3 we used T_{zz} , and in sec. 1.4 we used T_{zzz}. Since in our new notation $z = x_3$, we have by (49) and (50)

$$T_{zz} = T_{33} = \sum_{p,q=1}^{3} a_{p3} a_{q3} T'_{pq} , \tag{49'}$$

$$T_{zzz} = T_{333} = \sum_{p,q,r=1}^{3} a_{p3} a_{q3} a_{r3} T'_{pqr} , \tag{50'}$$

where a_{p3} is the cosine of the angle between the old x'_p-axis and the new x_3-axis, that is, the cosine of the angle between the local vertical and the old x'_p-axis.

2. Kinematical Measurement of Gravitational Gradients (Theoretical Background)

2.1. Separation of Gravitational and Inertial Effects

A gravimeter mounted aboard an airplane will measure the resultant of gravitational force and inertial acceleration due to irregular motion of the aircraft; the gravimeter will not be able to separate gravitational and inertial components. The reason is the equality of "gravitational mass" and "inertial mass," which has been established experimentally to an extremely high accuracy by Eötvös and others and explained by Einstein in his general theory of relativity. This is the Principle of Equivalence, according to which gravitation and inertia are essentially identical.

This principle seems to exclude any hope of separating gravitational and inertial effects. In fact, if the force vector at one particular point only is considered, this force cannot be separated into its gravitational and its inertial parts because of the identity of gravitational and inertial mass. Such a separation is, however, possible with higher-order gradients because of the structural differences between gravitational and inertial fields.

The practically simplest approach to this problem is in terms of classical mechanics (sec. 2.2), but this approach is not quite satisfactory because gravitational and inertial forces are treated as different concepts from the outset, which seems to be contrary to the Principle of Equivalence.

A deeper understanding requires a relativistic analysis (sec. 2.3). The general theory of relativity is considered the best theory of gravitation available

at present. It gives an explanation of the equality of gravitational and inertial mass by showing that these two kinds of mass are identical, whereas according to classical mechanics this equality is an accidental coincidence. Although the relativistic treatment will confirm the results of the classical mechanics, it is necessary in order to dispel the doubts raised by the Principle of Equivalence. In addition, the relativistic approach is quite appropriate to kinematical gravimetry because time, the kinematical element, is woven naturally into a unified space-time structure.

2.2. The Approach Through Classical Mechanics

It will be convenient to use a similar notation as in sec. 1.5 , denoting the coordinates x, y, z by x_1, x_2, x_3, or briefly by x_i, where i = 1, 2, 3.

Consider an <u>inertial coordinate system</u> X_i, that is, a coordinate system in which the Newtonian equations of motion

$$\ddot{X}_i = F_i \tag{51}$$

hold. A dot denotes differentiation with respect to time; therefore, $\ddot{X}_i$ will be the components of the vector of acceleration; F_i are the components of the force acting on a unit mass. Hence (51) is the well-known law: the force equals the product of mass (taken as unity) and acceleration.

Consider now another rectangular coordinate system x_i rigidly connected with the measuring apparatus. The systems $x_i = (x, y, z)$ and $X_i = (X, Y, Z)$ will be connected by an orthogonal coordinate transformation of the well-known form

$$X_i = \sum_{j=1}^{3} a_{ij} x_j + B_i \quad , \tag{52}$$

which differs from (45) only by B_i which denotes a displacement of the origin of the system x_i corresponding to the motion of the aircraft. Both the rotation matrix a_{ij} and the displacement vector B_i are functions of the time t.

We shall now introduce the familiar summation convention (e.g., James and James, 1959, p. 377), according to which the mere repetition of an index is sufficient to denote summation with respect to this index over its range. Hence (52) may be abbreviated as

$$X_i = a_{ij} x_j + B_i \quad , \tag{52'}$$

the summation being indicated by the repetition of the index j.

Repeated differentiation of (52') with respect to the time t yields

$$\begin{aligned} \dot{X}_i &= a_{ij}\dot{x}_j + \dot{a}_{ij} x_j + \dot{B}_i \quad , \\ \ddot{X}_i &= a_{ij}\ddot{x}_j + 2\dot{a}_{ij}\dot{x}_j + \ddot{a}_{ij} x_j + \ddot{B}_i \quad . \end{aligned} \tag{53}$$

Combining (51) and (53) we obtain

$$a_{ij}\ddot{x}_j = F_i - 2\dot{a}_{ij}\dot{x}_j - \ddot{a}_{ij} x_j - \ddot{B}_i \quad .$$

Multiplying by a_{ik} (the summation with respect to i is then automatically implied) we obtain

$$a_{ik} a_{ij} \ddot{x}_j = a_{ik} F_i - 2a_{ik}\dot{a}_{ij}\dot{x}_j - a_{ik}\ddot{a}_{ij} x_j - a_{ik}\ddot{B}_i \quad . \tag{54}$$

In agreement with (46) we have

$$a_{ik} a_{ij} = \delta_{kj} \tag{55}$$

(the summation may as well be over the first index i as over the second index as in (46)). Hence the left-hand side of (54) becomes

$$a_{ik} a_{ij} \ddot{x}_j = \delta_{kj}\ddot{x}_j = \ddot{x}_k \quad , \tag{56a}$$

since δ_{kj} is zero for $j \neq k$ and unity for $j = k$.

Multiplying (52') by a_{ik} we find

$$a_{ik}X_i = a_{ik}a_{ij}x_j + a_{ik}B_i \quad .$$

Just as in (56a) we have $a_{ik}a_{ij}x_j = x_k$, so that we obtain

$$x_k = a_{ik}X_i - b_k \tag{57}$$

with

$$b_k = a_{ik}B_i \quad .$$

Equation (57) represents the transformation inverse to (52') . From (57) we see that

$$f_k = a_{ik}F_i \tag{56b}$$

will denote the components of the force in the system x_i ; and similarly we put

$$\ddot{b}_k = a_{ik}\ddot{B}_i \quad , \tag{56c}$$

so that $\ddot{b}_k$ may be interpreted as the second time derivative of b_k with a_{ik} held constant.

By means of (56a, b, c) we reduce (54) to

$$\ddot{x}_k = f_k - 2a_{ik}\dot{a}_{ij}\dot{x}_j - a_{ik}\ddot{a}_{ij}x_j - \ddot{b}_k \quad . \tag{58}$$

This system may still be further transformed.

Differentiating (55) with respect to time, remembering that the right-hand side is identically constant, we find

$$\dot{a}_{ik}a_{ij} + a_{ik}\dot{a}_{ij} = 0 \quad , \tag{59}$$

so that the matrix

$$w_{jk} = a_{ik}\dot{a}_{ij} \tag{60}$$

is skew-symmetric:

$$w_{jk} = -w_{kj} \quad .$$

The matrix w_{jk} characterizes the instantaneous rotation since it may be shown that the vector

$$\vec{\omega} = (\omega_1, \omega_2, \omega_3) \ ;$$

$$\omega_1 = w_{23} \ , \quad \omega_2 = w_{31} \ , \quad \omega_3 = w_{12}$$

has the direction of the instantaneous axis of rotation (with respect to the moving system x_i); the magnitude of $\vec{\omega}$,

$$|\vec{\omega}| = \sqrt{w_{23}^2 + w_{31}^2 + w_{12}^2} = \omega \ ,$$

is the instantaneous angular velocity between the systems x_i and X_i .

By differentiating (60) we find

$$\dot{w}_{jk} = \dot{a}_{ik}\dot{a}_{ij} + a_{ik}\ddot{a}_{ij} \ ;$$

on the other hand,

$$w_{j\ell}w_{k\ell} = a_{m\ell}\dot{a}_{mj}a_{n\ell}\dot{a}_{nk} = \delta_{mn}\dot{a}_{mj}\dot{a}_{nk} = \dot{a}_{ij}\dot{a}_{ik} \ , \tag{61}$$

so that

$$a_{ik}\ddot{a}_{ij} = \dot{w}_{jk} - w_{j\ell}w_{k\ell} \ . \tag{62}$$

With (60) and (62), equation (58) goes over into

$$\ddot{x}_k = f_k - 2w_{jk}\dot{x}_j - (\dot{w}_{jk} - w_{j\ell}w_{k\ell})x_j - \ddot{b}_k$$

or, on replacing the index k by i and using (61),

$$\ddot{x}_i = f_i + 2w_{ij}\dot{x}_j + (\dot{w}_{ij} + w_{ik}w_{jk})x_j - \ddot{b}_i \ . \tag{63}$$

This is the final form of the equations of motion in the moving system x_i .[1] It is not of the Newtonian form (51). To put it into a Newtonian form,

$$\ddot{x}_i = f_i^* \ , \tag{64}$$

---

[1] For a more sophisticated derivation see (Morgenstern and Szabó, 1961, pp. 7-9).

we must set

$$f_i^* = f_i + 2w_{ij}\dot{x}_j + (w_{ik}w_{jk} + \dot{w}_{ij})x_j - \ddot{b}_i \quad , \tag{65}$$

thus adding to the "true" force f_i certain fictitious "inertial forces," of which

$$2w_{ij}\dot{x}_j$$

is the Coriolis force, and

$$w_{ik}w_{jk}x_j$$

is the centrifugal force.

If we consider f_i to be the gravitational force, it is the gradient vector of the gravitational potential V,

$$f_i = \frac{\partial V}{\partial x_i} \quad , \tag{66}$$

but the output of our measuring system, which operates according to the Newtonian form (64), will be f_i^*. By (65) and (66), with $\dot{x}_i = 0$ because the measuring system is at rest with respect to the frame xyz, we have

$$\frac{\partial V}{\partial x_i} = f_i^* - (w_{ik}w_{jk} + \dot{w}_{ij})x_j + \ddot{b}_i \quad . \tag{67}$$

This equation forms the starting point for our conclusions. Successive partial differentiations with respect to x_j yield

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_i^*}{\partial x_j} - (w_{ik}w_{jk} + \dot{w}_{ij}) \quad , \tag{68}$$

$$\frac{\partial^3 V}{\partial x_i \partial x_j \partial x_k} = \frac{\partial^2 f_i^*}{\partial x_j \partial x_k} \quad . \tag{69}$$

We shall now distinguish two cases:

1. <u>No inertial stabilization.</u> The measuring instrument is rigidly connected to the

aircraft and follows its linear acceleration and its rotational movements. Then both the gravitational vector, by (67), and the second derivatives, by (68), will falsified by inertial effects, but the third derivatives of the potential will no longer be affected by inertial disturbances, as (69) shows.

2. Inertial stabilization. Here the instrument axes xyz will be kept constantly parallel to inertial axes XYZ (to assume the simplest case) by inertial stabilization. Then

$$w_{ij} \equiv 0 ,$$

and (67) and (68) reduce to

$$\frac{\partial V}{\partial x_i} = f_i^* + \ddot{b}_i \quad , \tag{70}$$

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_i^*}{\partial x_j} \quad . \tag{71}$$

Here the gravitational vector, by (70), is still affected by linear accelerations, but inertial disturbances are absent already in the second derivatives (71).

Hence, by measuring higher derivatives of the potential, it is indeed possible to separate gravitational effects from inertial disturbances.[1]

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[1] Preliminary considerations along this line have been made by Balabushevich (1954); the separability in the higher derivatives was also recognized by Veselov (1964) through a different reasoning.

2.3. The Approach Through the General Theory of Relativity

2.3.1. Principles; the Kinematical Potentials

In the theory of relativity, space is combined with time into a space-time continuum.[1] The space-time of the special theory of relativity is a four dimensional manifold with coordinates

$$x_1 = x , \quad x_2 = y , \quad x_3 = z , \quad x_4 = t , \tag{72}$$

whose line element ds may be expressed as

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) ,$$

where c is the velocity of light in vacuum. For the purpose of the present paper it is more convenient to divide by the constant c^2, thus using for the line element the expression

$$ds^2 = dt^2 - c^{-2}(dx^2 + dy^2 + dz^2) . \tag{73}$$

This may also be written

$$ds^2 = dt^2 - c^{-2} dx_i dx_i , \tag{74}$$

where Latin indices denote the spatial indices 1, 2, 3 only; by (72) this expression is identical with (73), the summation convention being kept in mind.

---

[1] There are many excellent introductions to the theory of relativity. We mention (Bergmann, 1942), (Adler, Bazin, and Schiffer, 1965) and, more advanced but highly original, (Synge, 1964 and 1960); some more may be found in the list of references.

Now we shall introduce a further formal change, writing superscripts instead of subscripts:

$$ds^2 = dt^2 - c^{-2}dx^i dx^i \quad . \tag{74'}$$

Here dx^i is only another notation for dx_i ; this notation is in agreement with tensor calculus, where "covariant" indices are distinguished from "contravariant" indices, the former being written as subscripts, the latter as superscripts. Because the dx^i form a contravariant vector, the notation (74') is in agreement with tensor notation and therefore sometimes preferable.

The expression (74') has the general form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \quad , \tag{75}$$

if we include dt as dx^4 in agreement with (72) (note that the superscript 4 is not an exponent!) and let Greek indices run from 1 to 4, whereas, as we have mentioned, Latin indices run from 1 to 3 only. Because of the summation convention, (75) is, of course, equivalent to

$$ds^2 = \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} g_{\alpha\beta}dx^\alpha dx^\beta \quad .$$

The matrix $g_{\alpha\beta}$, the <u>covariant fundamental tensor</u>, has the form

$$(g_{\alpha\beta}) = \begin{pmatrix} -c^{-2} & 0 & 0 & 0 \\ 0 & c^{-2} & 0 & 0 \\ 0 & 0 & c^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{76}$$

as we recognize by comparing (75) and (74'). That is, we have

$$g_{44} = 1 \ ,$$

$$g_{i4} = g_{4i} = 0 \ , \tag{77}$$

$$g_{ij} = -c^{-2}\delta_{ij} \ .$$

The fundamental tensor $g_{\alpha\beta}$ has this simple diagonal form with constant coefficients only in an inertial coordinate system without gravitational forces. Such inertial coordinate systems in space-time are the four-dimensional analogue of rectangular coordinates in space.

If we use curvilinear coordinates in space-time, then the $g_{\alpha\beta}$ will no longer be equal to the constant values (77). This will happen with non-inertial coordinate systems such as accelerated or rotating frames: although the spatial 3-system is rectangular, it forms together with time a curvilinear 4-system. As an example, consider the rectangular non-inertial system $x_i = (x, y, z)$ of sec. 2.2. In the inertial system $X_i = (X, Y, Z)$, the fundamental tensor will have the simple form (76), so that

$$ds^2 = dt^2 - c^{-2}dX_i dX_i \ . \tag{78}$$

According to the transformation formula (52') we have

$$dX_i = a_{ij}dx_j + (\dot{a}_{ij}x_j + \dot{B}_i)\,dt \ .$$

Substituting this into (78) we obtain

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{79}$$

where now

$$g_{44} = 1 - c^{-2}(w_{j\ell}w_{k\ell}x_j x_k - 2w_{jk}\dot{b}_j x_k + \dot{b}_j\dot{b}_j) \ ,$$

$$g_{i4} = g_{4i} = c^{-2}(w_{ij}x_j - \dot{b}_i) \ , \tag{80}$$

$$g_{ij} = -c^{-2}\delta_{ij} \ .$$

Here we have used (55), (60), (61), and the substitution

$$\dot{b}_i = a_{ij} \dot{B}_i \quad . \tag{81}$$

As we have mentioned, the non-constant values (80) correspond to a curvilinear coordinate system in space-time. In the <u>general theory of relativity</u> Einstein showed that also a gravitational field can be represented by curvilinear coordinates in space-time, that is, by non-constant $g_{\alpha\beta}$. In this sense, gravitation and inertia are equivalent; this is the important <u>Principle of Equivalence</u>, which also explains the identity of inertial and gravitational mass.

In a system which as a whole is unaccelerated and nonrotating but which contains gravitational forces, the $g_{\alpha\beta}$ may be represented as power series with respect to c^{-2} , the inverse square of the velocity of light (Bergmann, 1942, p. 226):

$$\begin{aligned} g_{44} &= 1 + c^{-2}\underset{1}{h_{44}} + c^{-4}\underset{2}{h_{44}} + c^{-6}\underset{3}{h_{44}} + \cdots \ , \\ g_{4i} &= \qquad\qquad c^{-4}\underset{2}{h_{4i}} + c^{-6}\underset{3}{h_{4i}} + \cdots \ , \\ g_{ij} &= -c^{-2}\delta_{ij} + c^{-4}\underset{2}{h_{ij}} + c^{-6}\underset{3}{h_{ij}} + \cdots \ . \end{aligned} \tag{82}$$

Since in metric (c. g. s.) units, c^{-2} is an extremely small quantity, the different orders of magnitude are sharply discriminated in the expansion (82). The basic fact is that

$$\underset{1}{h_{44}} = -2V \ , \tag{83}$$

where V is the ordinary Newtonian potential of gravitation. That is,

$$V = k \int \frac{dm}{\ell} \ , \tag{84}$$

where k is the gravitational constant, dm is the element of mass, and ℓ is the distance between dm and the point to which V refers.

It may be mentioned that the $\underset{2}{h}_{ij}$ are also related to V by

$$\underset{2}{h}_{ij} = -2V\delta_{ij} .$$

This fact will not be used, however, because in this paper we shall consistently limit ourselves to the <u>linear approximation</u> (linear in c^{-2}), which at present is sufficient for almost all practical purposes:

$$\begin{aligned} g_{44} &= 1 - 2c^{-2}V , \\ g_{4i} &= 0 , \\ g_{ij} &= -c^{-2}\delta_{ij} . \end{aligned} \tag{85}$$

The fact that only the Newtonian potential V occurs in the linear approximation suggests that the results obtained by using (85) will be essentially equivalent to classical mechanics; and every textbook on general relativity shows that this is indeed true. Physically, the linear approximation corresponds to a slow motion (velocity <<c) in a weak gravitational field (such as that of the earth, the sun, etc.).

Let us now perform the transition to an accelerated and rotating frame. In exactly the same way as we transformed (77) into (80), we transform (85) into

$$\begin{aligned} g_{44} &= 1 - c^{-2}(2V + w_{j\ell}w_{k\ell}x_jx_k - 2w_{jk}\dot{b}_jx_k + \dot{b}_j\dot{b}_j) , \\ g_{i4} &= g_{4i} = c^{-2}(w_{ij}x_j - \dot{b}_i) , \\ g_{ij} &= -c^{-2}\delta_{ij} . \end{aligned} \tag{86}$$

To get a better understanding of these formulas, consider a system xyz connected with the rotating earth, the z-axis coinciding with the axis of rotation.

Then

$$b_i \equiv 0 \ , \tag{87a}$$

$$(w_{ij}) = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{87b}$$

so that (86) reduces to

$$\begin{aligned} g_{44} &= 1 - c^{-2}[2V + \omega^2(x^2 + y^2)] \ , \\ g_{14} &= c^{-2}\omega y \ , \\ g_{24} &= -c^{-2}\omega x \ , \\ g_{34} &= 0 \ , \\ g_{ij} &= -c^{-2}\delta_{ij} \ . \end{aligned} \tag{88}$$

Introducing the potential of gravity (gravitation plus centrifugal force),

$$W = V + \tfrac{1}{2}\omega^2(x^2 + y^2) \ , \tag{89}$$

we see at once that

$$g_{44} = 1 - 2c^{-2}W \ ; \tag{90}$$

that is, g_{44} is (apart from constants) <u>essentially identical with the potential of gravity</u>.

Similarly, if there is no rotation, (85) shows that g_{44} is essentially identical with the potential of gravitation, V. The fact that in (88) the potential of centrifugal force enters into g_{44} in a natural way, is another expression of the Principle of Equivalence, since gravitational and inertial forces occur in g_{44} side by side in an equivalent manner.

Thus in (88), g_{44} contains the statical components of the gravity field: gravitation and centrifugal force. It may be shown that the velocity-dependent component of the gravity field, the Coriolis force, is expressed by the quantities g_{i4}. We therefore infer that, also in the general case of (86), the complete set of $g_{\alpha\beta}$ will express all features of the combined gravitational-inertial field; the $g_{\alpha\beta}$ may therefore be called the kinematical gravity potentials.

Thus the extension of physical geodesy to moving systems requires the introduction of the complete set of the kinematical potentials $g_{\alpha\beta}$; the component g_{44} equivalent to the statical potential W is no longer sufficient to characterize the gravity field.

That the set of $g_{\alpha\beta}$ is necessary and sufficient for this purpose is expressed by the fact that the path of a particle under the combined influence of gravitational and inertial forces is a geodesic

$$\int ds = \int \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta} = \text{minimum} \quad . \tag{91}$$

The path is therefore determined by the $g_{\alpha\beta}$. We shall return to this point later, but we mention already here that the evaluation of (91) by means of the calculus of variations, using the $g_{\alpha\beta}$ as expressed by (86), leads just to the equations of motion (63) found previously (the restriction to the linear approximation being always understood).

Velocity, acceleration, force. Since space-time is four-dimensional, it is natural to define four-dimensional equivalents of terms such as velocity,

acceleration, or force. The 4-velocity is simply defined as

$$u^\alpha = \frac{dx^\alpha}{ds} \equiv \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dt}{ds}\right) . \tag{92}$$

From expressions such as (78), (80), or (82) we see that in every case

$$ds^2 = dt^2 + 0(c^{-2}) ,$$

where $0(c^{-2})$ denotes small terms of the order of c^{-2}. Hence also

$$ds = dt + 0(c^{-2}) , \tag{93}$$

so that

$$\frac{dt}{ds} = 1 + 0(c^{-2}) .$$

Neglecting a relative error of $0(c^{-2})$ in agreement with our linear approximation, we simply have

$$u^\alpha = \frac{dx^\alpha}{ds} = (\dot{x}, \dot{y}, \dot{z}, 1) + 0(c^{-2}) , \tag{94}$$

where, as usual, $\dot{x} = dx/dt$, etc. Hence, to the approximation considered, the spatial components of the 4-velocity u^α are the components of the ordinary 3-velocity

$$v^i = \dot{x}_i = (\dot{x}, \dot{y}, \dot{z}) ; \tag{95}$$

the temporal component is

$$u^4 = 1 .$$

The four-dimensional generalization of acceleration is less trivial. It would be obvious to define 4-acceleration by

$$\frac{du^\alpha}{ds} = \frac{d^2x^\alpha}{ds^2} ,$$

in analogy to (92), but it is shown in general tensor calculus that du^α/ds is not a

vector (as far as transformation properties are concerned). Instead, it is necessary to define the vector of 4-acceleration by

$$a^{\alpha} = \frac{\delta u^{\alpha}}{\delta s} = \frac{\delta^2 x^{\alpha}}{\delta s^2} , \tag{96}$$

where $\delta u^{\alpha}/\delta s$ is the "covariant derivative"

$$\frac{\delta u^{\alpha}}{\delta s} = \frac{du^{\alpha}}{ds} + \Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma} , \tag{97}$$

the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ being given by

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} + \frac{\partial g_{\gamma\delta}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}} \right) , \tag{98}$$

where the matrix $(g^{\alpha\beta})$, the contravariant fundamental tensor, is inverse to the matrix $(g_{\alpha\beta})$.

The second term on the right-hand side of (97) is of decisive importance, because it expresses the gravitational and inertial forces. In fact, we have (always in the linear approximation), for the inertial system, the $g_{\alpha\beta}$ defined by (77):

$$\begin{aligned} a^i &= \ddot{x}_i , \\ a^4 &= 0 ; \end{aligned} \tag{99a}$$

for the purely gravitational system (85) :

$$\begin{aligned} a^i &= \ddot{x}_i - \frac{\partial V}{\partial x_i} , \\ a^4 &= 0 ; \end{aligned} \tag{99b}$$

for the non-inertial gravitational system (86):

$$\begin{aligned} a^i &= \ddot{x}_i - \frac{\partial V}{\partial x_i} - 2w_{ij}\dot{x}_j - (\dot{w}_{ij} + w_{ik}w_{jk})x_j + \ddot{b}_i , \\ a^4 &= 0 ; \end{aligned} \tag{99c}$$

here we have used subscripts instead of superscripts for the x and their derivatives in order to be in agreement with the notation of sec. 2.2.

A geodesic is a line along which the 4-acceleration is zero:

$$a^{\alpha} = 0 \ . \tag{100}$$

In fact, the evaluation of (91) by the calculus of variations leads to the equation

$$\frac{d^2x^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{ds} \frac{dx^{\gamma}}{ds} = 0 \ , \tag{101}$$

which is identical with (100), as the comparison of (96) and (97) shows.

By (99 a, b, c), the formally simple condition (100) leads indeed to the correct equations of motion of a particle subject only to gravitational and inertial forces (if present), but free otherwise.

The four-dimensional generalization of Newton's fundamental law, force equals the product of mass and acceleration, will be

$$f^{\alpha} = m \ a^{\alpha} \ ; \tag{102}$$

but it should be noted that all gravitational (and inertial) forces are to be excluded from the 4-force f^{α}. As usual, we put the mass m equal to unity, so that

$$f^{\alpha} = a^{\alpha} = \frac{\delta u^{\alpha}}{\delta s} \ . \tag{102'}$$

Consider now the case (99c), corresponding to the moving measuring apparatus of sec. 2.2. Here we must put $\ddot{x}_i = 0 = \dot{x}_i$, because the measuring system is at rest with respect to the frame xyz. Then from (99c) and (102') we have

$$\begin{aligned} f^{i} &= -\frac{\partial V}{\partial x_i} - (\dot{w}_{ij} + w_{ik} w_{jk}) x_j + \ddot{b}_i \ , \\ f^{4} &= 0 \ . \end{aligned} \tag{103}$$

Comparing this with (67) we see that

$$f^i = - f_i^* ; \qquad (104)$$

that is, apart from the sign, the spatial components of the 4-force f^α are identical with the apparent force f_i^* of sec. 2.2. The difference in sign expresses the principle of action and reaction: f_i^* is defined as the acting force, whereas f^i is the force of reaction arising from the measuring system being forced to remain at rest in the frame xyz.

The agreement between classical and relativistic analysis, as expressed in (104), was to be expected for the linear approximation, because Einstein's theory goes over into Newton's theory for weak fields and low velocities. To a certain extent, this fact confirms the results obtained at the end of sec. 2.2 , because with (104) also the validity of (69) is verified. As for the case of inertial stabilization, leading to (71), the possibility of such a stabilization in the presence of a gravity field ought to be investigated by means of the general theory of relativity. This will be done in sec. 2.3.3 ; an inherently relativistic method will then provide a final justification in sec. 2.3.4.

2.3.2. Gravitational Gradients and the Riemannian Tensor

Let us start with a simple example familiar to the geodesist. Consider rectangular coordinates x, y in the plane; then the line element will be given by

$$ds^2 = dx^2 + dy^2 \quad . \qquad (105)$$

On introducing curvilinear coordinates u, v in the plane by

$$\begin{aligned} x &= x(u, v) \quad , \\ y &= y(u, v) \quad , \end{aligned} \qquad (106)$$

the line element (105) takes the form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \quad , \tag{107}$$

where E, F, G are functions of u and v.

The line element on an arbitrary curved surface has the same form (107). We may therefore ask by which criterion one is able to recognize whether a line element of the form (107) refers to a curved surface or to a plane. The answer (due to Gauss) is as follows: Form a certain expression containing E, F, G and its first and second derivatives; this expression is called Gaussian curvature. If the Gaussian curvature is zero, then (107) refers to a plane and can be transformed into the simple form (105); if the Gaussian curvature is nonzero, then (107) refers to a curved surface and cannot be transformed (over a finite area) into (105).

The remarkable fact is that the principle of Einstein's theory of gravitation is the extension of two-dimensional Gaussian surface theory to four-dimensional space-time; this fact should be particularly appealing to the geodesist.

The flat space-time of special relativity corresponds to the plane; in fact, by substituting

$$ic^{-1}x = \xi \ , \quad ic^{-1}y = \eta \ , \quad ic^{-1}z = \zeta \ ,$$

the line element (73) goes over into

$$ds^2 = d\xi^2 + d\eta^2 + d\zeta^2 + dt^2 \quad , \tag{108}$$

which is the precise four-dimensional equivalent of (105). Thus the inertial systems of special relativity correspond to the rectangular coordinate systems in the plane. Transformations to accelerated and rotating frames after the fashion

of (52') correspond to the transition to curvilinear coordinates in the plane (equations (106)), and the resulting expression (79) is the four-dimensional equivalent of (107); in fact, in two dimensions,

$$g_{11} = E, \quad g_{12} = g_{21} = F, \quad g_{22} = G. \tag{109}$$

A transition to "curvilinear coordinates" (accelerated and rotating frames) in space-time without gravitation does not change its flat character, in the same way as the plane, even when expressed in curvilinear coordinates, still remains a plane. Thus the $g_{\alpha\beta}$ of (80) still refer to flat space, although inertial forces occur.

The $g_{\alpha\beta}$ of (82), however, which express a true gravitational field, are of a completely different character. They are comparable to the E, F, G of a curved surface. Gravitation corresponds to a curved space-time.

In surface theory the criterion of flatness is the Gaussian curvature. Its four-dimensional generalization is the Riemannian curvature tensor, a set of 20 independent quantities. If all these quantities are zero, if

$$R_{\alpha\beta\gamma\delta} = 0 , \tag{110}$$

there is no gravitation, no matter how complicated the $g_{\alpha\beta}$ look; in this case there are only inertial forces.

Gravitation gives rise to a curved space-time and thus to a nonzero Riemannian tensor; we may write symbolically (Synge, 1960, p. 109) :

$$R_{\alpha\beta\gamma\delta} = \text{gravitational field.} \tag{111}$$

The analytical expression of $R_{\alpha\beta\gamma\delta}$ contains the $g_{\alpha\beta}$ and its first and second derivatives; we have

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 g_{\beta\gamma}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 g_{\beta\delta}}{\partial x^\alpha \partial x^\gamma}\right) +$$

$$+ g_{\mu\nu}\left(\Gamma^\mu_{\alpha\delta}\Gamma^\nu_{\beta\gamma} - \Gamma^\mu_{\alpha\gamma}\Gamma^\nu_{\beta\delta}\right) , \tag{112}$$

the Christoffel symbols being defined by (98) .

Since $R_{\alpha\beta\gamma\delta}$ contains second derivatives of $g_{\alpha\beta}$, we cannot hope to be able to separate gravitational and inertial effects using the potential ($\sim g_{44}$) or its first derivatives, that is, the force vector, only. Separation, if possible at all, can only occur with second or higher derivatives; this is in agreement with the results of sec. 2.2 obtained from classical mechanics. This is another confirmation of the importance of second (and possibly higher) derivatives in kinematical geodesy.

The individual components of the Riemannian tensor show a clear distinction as to order of magnitude, the differences being of the order of c^{-2}. Using (82) with (112) we find

$$\begin{aligned} c^2 R_{i4j4} &= -\frac{1}{2}\frac{\partial^2 \overset{1}{h}_{44}}{\partial x_i\, \partial x_j} + 0(c^{-2}) , \\ c^2 R_{ijk\ell} &= 0(c^{-2}) , \\ c^2 R_{ijk4} &= 0(c^{-2}) ; \end{aligned} \tag{113}$$

the other components are either expressible through these or are zero (e.g., Synge, 1960, p. 16). The Latin subscripts denote spatial indices, as usual.

We see that $R_{ijk\ell}$ and R_{ijk4} are smaller than R_{i4j4} by the factor c^{-2} . Hence, neglecting $0(c^{-2})$ as usual, we have

$$c^2 R_{ijk\ell} = 0 \ ,$$

$$c^2 R_{ijk4} = 0 \ ,$$

so that the only surviving equation of (113) is the first; with (83) we have

$$c^2 R_{i4j4} = \frac{\partial^2 V}{\partial x_i \, \partial x_j} \ . \tag{114}$$

This simple equation shows that the component R_{i4j4} of the Riemannian curvature tensor is essentially equivalent to a second derivative of the gravitational potential V. This will be extremely important for the following sections.

If we had used the $g_{\alpha\beta}$ of (86) instead of (85) we would have obtained the same result (114) since the contribution of the inertial forces to the Riemannian tensor is zero.

2.3.3. Gyroscopic Stabilization and Fermi Propagation

At the end of sec. 2.2 we have seen that gravitational effects are separated from inertial disturbances already in the second derivatives if the instrumental axes xyz are kept constantly parallel to the inertial axes XYZ.

This result is consistent with the relativistic approach of sec. 2.3.1 , but the method of keeping the instrumental axes permanently parallel must be examined. The direction of the instrumental axes is kept constant by inertial stabilization by means of gyroscopes. The principle is that the axis of a freely spinning gyroscope maintains its direction even if its frame is accelerated or rotated; furthermore, the axis is unaffected by gravity.

It is by no means obvious that this is true also in the general theory of relativity; one should even be inclined to suspect the contrary. Therefore, this

problem deserves closer examination.

First of all, the question arises as to whether something like constancy of spatial direction, or briefly, a nonrotating frame, can be defined within general relativity. Surprisingly enough, this is possible even in a local manner (an example to the contrary: uniform motion of a mass point cannot be defined in an intrinsic local way). The relevant concept is Fermi propagation, which is considered in detail and used extensively in (Synge, 1960). In view of the importance of Fermi propagation, it is surprising that this concept is not even mentioned in any other standard text on relativity known to the author.[1]

Synge interpreted Fermi propagation physically by means of an idealized optical experiment ("the bouncing photon"). Pirani (1956) pointed out that the axis of an idealized gyroscope ("spinning test particle") also follows Fermi propagation. To show this, he used Papapetrou's (1951) equation of motion of the spin of a spinning test particle. Papapetrou's (1951) developments refer to a free test particle; he later (Papapetrou and Urich, 1955) generalized them to a spinning particle moving in an electromagnetic field and found the same spin equations. For our present purpose, where we have a free gyroscope whose frame performs a constrained arbitrary motion, the case of a freely spinning test particle under the influence of external forces would be relevant. This case can be handled by an extension of a method of (Weber, 1964). The result is that

[1] The "successive Lorentz transformations" of (Møller, 1952, §§ 46 and 96) are essentially equivalent to Fermi transport specialized to flat space-time.

the equation of motion of the spin is the same as for a free particle; therefore, Fermi propagation holds also for the axis of a free gyro whose frame performs an arbitrary constrained motion. See also (Bertotti, 1962, p. 184) and (Bertotti, Brill, and Krotkov, 1962).

The equation of Fermi propagation may be written as

$$\frac{\delta\lambda^\alpha}{\delta s} = \lambda_\beta \left(\frac{\delta u^\beta}{\delta s} u^\alpha - \frac{\delta u^\alpha}{\delta s} u^\beta \right) \tag{115}$$

(Synge, 1960, p. 13). Here λ^α (or λ_β) are the contravariant (or covariant) components of the vector undergoing Fermi propagation. The vector u^α is the 4-velocity (92), the unit vector of the tangent to the particle's "world line." Its covariant derivative is given by (97), and the covariant derivative of λ^α is defined accordingly by

$$\frac{\delta\lambda^\alpha}{\delta s} = \frac{d\lambda^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}\lambda^\beta u^\gamma \ . \tag{116}$$

In our case, the vector λ^α represents the spin axis of the gyro. It lies in the instantaneous 3-space of the spinning particle and is consequently orthogonal to u^α:

$$u^\alpha \lambda_\alpha = 0 \ . \tag{117}$$

For space-like vectors satisfying (117) the general equation (115) reduces to

$$\frac{\delta\lambda^\alpha}{\delta s} = \lambda_\beta \frac{\delta u^\beta}{\delta s} u^\alpha \ . \tag{118}$$

This equation expresses the fact that the change $\delta\lambda^\alpha/\delta s$ has the direction of u^α and consequently no component in the instantaneous 3-space. This fact already suggests that Fermi propagation is related to spatial parallelism.

Consider now a system of three orthogonal vectors λ^{β}, each of them being represented by the axis of a freely spinning gyroscope. In this way the axes of a rectangular xyz system in space may be realized physically. Will now the respective coordinate axes so defined remain parallel to each other in the ordinary sense if the frame of the three gyros is moved arbitrarily in space, corresponding to the motion of the carrier aircraft?

The answer will be obtained by applying the linear approximation (85) to Fermi propagation. For the spatial components λ^{i}, in which we are interested, we may write (118) as

$$\frac{\delta\lambda^{i}}{\delta s} = \Phi u^{i} \tag{119}$$

where the factor Φ is given by

$$\Phi = \lambda_{\beta}\frac{\delta u^{\beta}}{\delta s} = g_{\alpha\beta}\lambda^{\alpha}\frac{\delta u^{\beta}}{\delta s} \quad . \tag{120}$$

In the global coordinate system xyzt of (85) the series expansion of the vector λ^{α} in the fashion of (82) reads

$$\begin{aligned} \lambda^{i} &= \underset{0}{\lambda}{}^{i} + c^{-2}\underset{1}{\lambda}{}^{i} + c^{-4}\underset{2}{\lambda}{}^{i} + \cdots \, , \\ \lambda^{4} &= \qquad\; c^{-2}\underset{1}{\lambda}{}^{4} + c^{-4}\underset{2}{\lambda}{}^{4} + \cdots \, , \end{aligned} \tag{121}$$

because a non-zero $\underset{0}{\lambda}{}^{4}$ would be incompatible with the orthogonality condition (117). We further have, in agreement with definition (96),

$$\begin{aligned} \frac{\delta u^{i}}{\delta s} &= a^{i} = \underset{0}{a}{}^{i} + c^{-2}\underset{1}{a}{}^{i} + \cdots \, , \\ \frac{\delta u^{4}}{\delta s} &= a^{4} = \qquad\; c^{-2}\underset{1}{a}{}^{4} + \cdots \, , \end{aligned} \tag{122}$$

because $\underset{0}{a}^4 = 0$ from (99b) .

We now insert (85), (121), and (122) into (120). The result is

$$\Phi = (1 - 2c^{-2}V)\, c^{-4} \underset{1}{\lambda}^4 \underset{1}{a}^4 - c^{-2} \underset{0}{\lambda}^i \underset{0}{a}^i + 0(c^{-4}) = 0(c^{-2}) \; .$$

Thus from (119) we have

$$\frac{\delta \lambda^i}{\delta s} = 0(c^{-2}) \; ; \tag{123}$$

that is, the change of λ^i (the gyro axis) is of the order of c^{-2} and may therefore be safely neglected.

Thus, general relativity confirms the fact that gyroscopic stabilization of instrumental axes is indeed possible; the effect of the gravity field is negligible in this case.

2.3.4. Application of Synge's Method

Synge (1960, pp. 156-158) has given a general theory of a "relativistically valid geodetic survey" to determine the Riemann tensor, which expresses the gravitational field. His formula is

$$R_{(abcd)} = g_{\alpha\beta} X^{\alpha}_{(a)} \left(\frac{\delta^2 X^{\beta}_{(b)}}{\delta y_{(c)} \delta y_{(d)}} - \frac{\delta^2 X^{\beta}_{(b)}}{\delta y_{(d)} \delta y_{(c)}} \right) . \tag{124}$$

The subscripts (a), (b), (c), (d) assume the values 1, 2, 3, 4 each. The coordinates x^{α} are expressed in terms of four parameters $y_{(a)}$ by equations of the form

$$x^{\alpha} = x^{\alpha} (y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}) \; ; \tag{125}$$

then

$$X^{\alpha}_{(a)} = \frac{\partial x^{\alpha}}{\partial y_{(a)}} \tag{126}$$

form a tetrad to which the components $R_{(abcd)}$ refer.

We shall now specialize the parameters in such a way that the base vectors of the tetrad become

$$X^{\alpha}_{(i)} = \lambda^{\alpha}_{(i)} \quad \text{for} \quad i = 1, 2, 3, \; ; \tag{127a}$$

$$X^{\alpha}_{(4)} = \lambda^{\alpha}_{(4)} = u^{\alpha} \; . \tag{127b}$$

Here $\lambda^{\alpha}_{(1)}$, $\lambda^{\alpha}_{(2)}$, $\lambda^{\alpha}_{(3)}$, $\lambda^{\alpha}_{(4)}$ are to form a system of four orthogonal unit vectors, the first three $\lambda^{\alpha}_{(i)}$ lying in the instantaneous 3-space of the moving particle, and the "time-like" unit vector $\lambda^{\alpha}_{(4)}$ is identical with the 4-velocity u^{α} tangent to the world line of the moving apparatus. The vectors $\lambda^{\alpha}_{(1)}$, $\lambda^{\alpha}_{(2)}$, $\lambda^{\alpha}_{(3)}$ are propagated along the world line by Fermi transport. The $R_{(abcd)}$ are the components of the Riemannian curvature tensor in the local system of the four unit vectors $\lambda^{\alpha}_{(a)}$.

By comparing (126) and (127a) we see that

$$\frac{\partial x^{\alpha}}{\partial y_{(4)}} = X^{\alpha}_{(4)} = \lambda^{\alpha}_{(4)} = u^{\alpha} = \frac{dx^{\alpha}}{ds} \; ,$$

so that along the world line under consideration we have (the point $s = 0$ being suitably chosen)

$$y_{(4)} = s \; , \tag{128}$$

and by (93)

$$y_{(4)} = t + 0(c^{-2}) \; . \tag{129a}$$

Similarly we may take

$$\begin{aligned} y_{(1)} &= x + 0(c^{-2}) \; , \\ y_{(2)} &= y + 0(c^{-2}) \; , \\ y_{(3)} &= z + 0(c^{-2}) \; ; \end{aligned} \tag{129b}$$

according to (123), the change of the spatial $\lambda^{i}_{(a)}$ is only of the order of c^{-2}, so that to this accuracy they may be identified with the unit vectors of three-dimensional inertial coordinates; therefore the identification (129b) is permissible.

With the measuring accuracy possible at present, it is best to concentrate on the components $R_{(i4j4)}$ of the curvature tensor, where i, j = 1, 2, 3 ; by (113) these components are predominant by the order of c^2.

The following derivations are rigorous; the approximations (129a, b) will be introduced only at the final stage. By (124), (127b) and (128) we have along the world line

$$R_{(i4j4)} = g_{\alpha\beta}\lambda^{\alpha}_{(i)} \left(\frac{\delta^2 u^\beta}{\delta y_{(j)}\delta s} - \frac{\delta^2 u^\beta}{\delta s \delta y_{(j)}} \right) . \tag{130}$$

Introducing the 4-force f^α by (102') we obtain

$$R_{(i4j4)} = g_{\alpha\beta}\lambda^{\alpha}_{(i)} \frac{\delta f^\beta}{\delta y_{(j)}} - g_{\alpha\beta}\lambda^{\alpha}_{(i)} \frac{\delta^2 u^\beta}{\delta s \delta y_{(j)}} . \tag{130'}$$

We shall now prove the relation

$$\frac{\delta u^\beta}{\delta y_{(j)}} = \frac{\delta \lambda^{\beta}_{(j)}}{\delta s} . \tag{131}$$

By the definition of the covariant derivative we have

$$\frac{\delta u^\beta}{\delta y_{(j)}} = \frac{\partial u^\beta}{\partial y_{(j)}} + \Gamma^{\beta}_{\gamma\delta} u^\gamma \frac{\partial x^\delta}{\partial y_{(j)}} = \frac{\partial^2 x^\beta}{\partial y_{(j)}\partial s} + \Gamma^{\beta}_{\gamma\delta} u^\gamma \lambda^{\delta}_{(j)} ,$$

$$\frac{\delta \lambda^{\beta}_{(j)}}{\delta s} = \frac{\partial \lambda^{\beta}_{(j)}}{\partial s} + \Gamma^{\beta}_{\gamma\delta}\lambda^{\gamma}_{(j)} \frac{\partial x^\delta}{\partial s} = \frac{\partial^2 x^\beta}{\partial s \partial y_{(j)}} + \Gamma^{\beta}_{\gamma\delta}\lambda^{\gamma}_{(j)} u^\delta .$$

Because the order of two consecutive partial differentiations can be interchanged, and because $\Gamma^{\beta}_{\gamma\delta}$ is symmetric with respect to γ and δ, these two expressions are identical, and (131) has been proved.

Hence

$$\frac{\delta^2 u^\beta}{\delta s \delta y_{(j)}} = \frac{\delta^2 \lambda^\beta_{(j)}}{\delta s^2} \quad . \tag{132}$$

The second derivative on the right-hand side may be evaluated by twice applying the formulas for Fermi propagation; for convenience we drop the subscript (j). First (118) gives

$$\frac{\delta \lambda^\beta}{\delta\, s} = g_{\gamma\delta} \lambda^\delta \frac{\delta u^\gamma}{\delta s} u^\beta$$

or, by (102') ,

$$\frac{\delta \lambda^\beta}{\delta s} = u^\beta g_{\mu\nu} f^\mu \lambda^\nu \ . \tag{133}$$

Writing (115) for an arbitrary vector L^β we have

$$\frac{\delta L^\beta}{\delta s} = g_{\gamma\delta} L^\delta \left(\frac{\delta u^\gamma}{\delta s} u^\beta - \frac{\delta u^\beta}{\delta s} u^\gamma \right) .$$

Substituting for L^β the expression (133) gives

$$\frac{\delta}{\delta s} \left(\frac{\delta \lambda^\beta}{\delta s} \right) = \frac{\delta^2 \lambda^\beta}{\delta s^2} = g_{\gamma\delta} u^\delta g_{\mu\nu} f^\mu \lambda^\nu (f^\gamma u^\beta - f^\beta u^\gamma) \ ,$$

again using (102'). Since u^γ is a time-like unit vector we have (Synge, 1960, p. 2)

$$g_{\gamma\delta} u^\gamma u^\delta = -1 \tag{134a}$$

and on differentiation with (102')

$$g_{\gamma\delta} f^\gamma u^\delta = 0 \ . \tag{134b}$$

By means of these equations we finally obtain

$$\frac{\delta^2\lambda^\beta}{\delta s^2} = f^\beta g_{\mu\nu} f^\mu \lambda^\nu . \tag{135}$$

Hence (132) becomes

$$\frac{\delta^2 u^\beta}{\delta s \delta y_{(j)}} = f^\beta g_{\mu\nu} f^\mu \lambda^\nu_{(j)} ,$$

so that (130') goes over into

$$R_{(i4j4)} = g_{\alpha\beta}\lambda^\alpha_{(i)} \frac{\delta f^\beta}{\delta y_{(j)}} - g_{\alpha\beta}\lambda^\alpha_{(i)} f^\beta g_{\mu\nu}\lambda^\mu_{(j)} f^\nu . \tag{136}$$

We now have (Synge, 1960, p. 10)

$$f_\alpha \lambda^\alpha_{(i)} = g_{\alpha\beta}\lambda^\alpha_{(i)} f^\beta = f_{(i)} , \tag{137}$$

the "covariant" components of f^α with respect to the orthonormal triad $\lambda^\alpha_{(i)}$.

By differentiation of (137) we find

$$\frac{\partial f_{(i)}}{\partial y_{(j)}} = \frac{\delta g_{\alpha\beta}}{\delta y_{(j)}} \lambda^\alpha_{(i)} f^\beta + g_{\alpha\beta} \frac{\delta \lambda^\alpha_{(i)}}{\delta y_{(j)}} f^\beta + g_{\alpha\beta}\lambda^\alpha_{(i)} \frac{\delta f^\beta}{\delta y_{(j)}} .$$

The first term on the right-hand side is zero because the covariant derivative of the fundamental tensor $g_{\alpha\beta}$ is identically zero; the second term can be made zero by a suitable choice of parameters $y_{(i)}$, and the third term is identical with the first term on the right-hand side of (136). Thus (136) becomes, (137) also being considered,

$$R_{(i4j4)} = \frac{\partial f_{(i)}}{\partial y_{(j)}} - f_{(i)} f_{(j)} . \tag{138}$$

This simple result is rigorous. Still using no approximation, it is appropriate to go over to upper ("contravariant") indices by

$$f_{(i)} = g_{(i\alpha)} f^{(\alpha)} ,$$

where $g_{(i\alpha)}$ is the matrix (76); cf. (Synge, 1960, p. 9). Since

$$f^{(4)} = f_{\alpha} u^{\alpha} = 0$$

by (134b), we find

$$f_{(i)} = - c^{-2} f^{(i)} . \tag{139}$$

Thus (138) becomes

$$c^2 R_{(i4j4)} = - \frac{\partial f^{(i)}}{\partial y_{(j)}} - c^{-2} f^{(i)} f^{(j)} . \tag{140}$$

This formula, slightly less simple than (138) and as rigorous, is more instructive because $f^{(i)}$ is equivalent to an ordinary 3-force also with respect to magnitude, and thus the orders of magnitude are immediately obvious in (140).

Now we shall finally proceed to our usual linear approximation. Evidently the second term on the right-hand side of (140) can be neglected, and there remains

$$c^2 R_{(i4j4)} = - \frac{\partial f^{(i)}}{\partial y_{(j)}} . \tag{140'}$$

Conformably we may use the identification (129b) ,

$$y_{(i)} = x_i + 0(c^{-2}) ,$$

so that

$$f^{(i)} = f^i + 0(c^{-2}) ,$$

$$R_{(i4j4)} = R_{i4j4} + 0(c^{-2}) .$$

Thus (140') reduces in the linear approximation to

$$c^2 R_{i4j4} = - \frac{\partial f^i}{\partial x_j} . \tag{141}$$

This equation expresses certain components of the Riemannian curvature tensor in terms of the measurable gradient of the 3-force f^i. By (114), which reads

$$c^2 R_{i4j4} = \frac{\partial^2 V}{\partial x_i \, \partial x_j} ,$$

these components are connected with the second derivatives of the gravitational potential V. Thus we obtain by comparing these two equations:

$$\frac{\partial^2 V}{\partial x_i \, \partial x_j} = - \frac{\partial f^i}{\partial x_j} . \tag{142}$$

This equation is identical with (71) because of (104), <u>but was obtained by a strictly relativistic argument</u>. This result is of importance because it provides the final justification of the approach through classical mechanics. The effect of the inertial forces is indeed removed if second derivatives of the potential are measured by means of a gyroscopically stabilized apparatus.

2.3.5. Relativistic Conclusions

The analysis of our problem by means of the general theory of relativity has confirmed the result of classical mechanics that gravitational and inertial forces can in fact be separated, at least to a very good approximation, in the second and higher derivatives of the potential.

On the one hand, this is to be expected because Einstein's theory of gravitation goes over, as a limit for a weak gravitational field and low velocities, into Newton's theory. On the other hand, there seems to be a disagreement with basic principles of general relativity, in particular with the Principle of Equivalence, according to which gravitational and inertial forces are basically identical. The obvious conclusion from this principle is that these two kinds of forces are indistinguishable and cannot, therefore, be separated.

Our results are also in apparent disagreement with the Principle of General Covariance, according to which all coordinate systems (inertial and noninertial, unaccelerated and accelerated, nonrotating and rotating) are equivalent in principle; there are no "privileged" coordinate systems (such as inertial systems) when gravitation is present. How can we explain these contradictions?

Both the Principle of Equivalence and the Principle of General Covariance have played a fundamental heuristic role in Einstein's considerations leading to his theory of gravitation around 1915, because these principles provide a natural and mathematically obvious transition from the flat space-time of special relativity to the curved space-time of general relativity. Since Einstein's heuristic procedure, although it is not a logical deduction, possesses great intuitive force, it is strongly emphasized in almost every introductory textbook on general relativity, subtler distinctions being usually disregarded or bypassed.

Our present problem, however, requires precisely these subtler distinctions. We shall briefly present our point, which is in essential conformity with Fock (1959) and Synge (1960).

We start with the Principle of General Covariance. The application of the relativistic theory of gravitation to the region of our solar system requires boundary conditions at infinity: with increasing distance from the attracting masses the effect of gravitation vanishes, and the curved space-time becomes flat at infinity. This fact permits the introduction of uniquely defined privileged coordinate systems, the harmonic coordinate systems, which to a certain extent correspond to the inertial coordinate systems of special relativity (Fock, 1959).

Harmonic (or "isothermic") coordinates date back to de Donder in 1921 and Lanczos in 1922; their geometrical and physical significance is explained in (Darmois, 1927) and (Levi-Civita, 1950), and their uniqueness in connection with the boundary conditions at infinity has been investigated by Fock (1959), who used them consistently and strongly emphasized (and perhaps overemphasized) their significance.

General covariance is an extremely efficient means of expressing physical laws as general theorems of the geometry of space-time by means of the tensor calculus. It does not really assert, however, that in some cases physical laws may not assume a simpler form in certain privileged coordinate systems. The gravitational equations do assume a simpler form when a harmonic coordinate

system is used; for instance, such a "quasi-inertial" system underlies the expansion (82).

As to the Principle of Equivalence, it is undoubtedly true (because of the identity of gravitational and inertial mass) as long as the force acting at one point only is considered. As soon as we consider a region in space, even an arbitrarily small one, however, we have an objective criterion as to whether a true gravitational field is present or not. For then we can form the Riemannian tensor by means of the forces and their first derivatives (the derivatives being obtained if the forces are known in an arbitrarily small region); cf. equation (138). If the Riemannian tensor vanishes, there is no gravitation; if it is different from zero, there is a true gravitational field. For careful expositions of the Principle of Equivalence see (Eddington, 1924, pp. 39-41) and (Bergmann, 1962, pp. 204-206).

The Riemannian curvature provides a criterion for the presence of a gravitational field, but not yet a means for the separation of gravitational and inertial effects. In flat space-time inertial forces have an objective significance because they are due to the deviation of the observer's coordinate system from an inertial system, which represents a Cartesian coordinate system in space-time. In the presence of a gravitational field, however, space-time is irregularly curved, and the use of global Cartesian coordinates is no longer possible.

Thus a separation of gravitational and inertial effects is feasible only if we succeed in introducing a privileged coordinate system sharing many desirable properties of Cartesian coordinates, and this separation is possible only to the extent to which the privileged coordinate system has the properties of Cartesian coordinates (it cannot have all of them because space-time is irregularly curved). Probably the best choice for such privileged "quasi-inertial" coordinate systems are the harmonic coordinate systems mentioned above. Inertial forces, by definition, will then occur when the observer's coordinate system differs from a harmonic system.

This is particularly evident in the linear approximation, because then the gravitational field can be assumed to be embedded in a flat space, and the harmonic coordinate systems can be identified with the inertial coordinate systems of this auxiliary flat space. In fact, gravitational and inertial force fields are <u>superposed linearly</u> without interaction; compare equations (77) (flat space), (80) (pure inertial force field), (85) (pure gravitational field), and (86) (combined field of gravitational and inertial forces).

Within the linear approximation, inertial systems (or rather three axes parallel to inertial axes in space) may be defined even locally; they coincide with global systems to an accuracy of $0(c^{-2})$, as we have seen in sec. 2.3.3 ; according to Eddington (1924, p. 99), a local terrestrial inertial coordinate system will deviate from a global inertial system by about 2 seconds of arc in a century.

Whereas geodesics, in general, by no means appear as straight lines in three-dimensional space (elliptical planetary orbits are geodesics in four-dimensional space-time!), it may be shown that null geodesics representing light rays in vacuum do appear as three-dimensional straight lines (to $0(c^{-2})$; the bending of a light ray grazing the limb of the sun is a higher-order effect).

Also, by (93), "proper time" s coincides to $0(c^{-2})$ with "coordinate time" t. Hence, even in the "general theory of relativity" there are at least three "absolute" concepts in the linear approximation: time, straight lines in space (represented by light rays), and rotation; these concepts are absolute in the sense that they are practically the same whether viewed globally or locally, so that they can be determined even by local measurements.

The deeper reason for the existence of the linear approximation is that gravitational and inertial effects show a characteristic hierarchy of orders of magnitude, the differences being of the order of c^{-2} and therefore very marked.

We may summarize the three main reasons for the possibility of separation of gravitational and inertial effects as follows:

1. The Riemannian curvature tensor provides a means of locally detecting the presence of a gravitational field.
2. Harmonic coordinate systems serve as "quasi-inertial" coordinate systems, with respect to which inertial forces may be defined.
3. Sharply distinguished orders of magnitude greatly facilitate the separation.

The linear approximation, which is fully sufficient for our present purpose, makes our separation practically unambiguous; however, in more exotic gravity fields considered in general relativity, where harmonic coordinates that go over into inertial Cartesian coordinates at infinity may not even exists, a separation of gravitational and inertial effects may well be meaningless.

3. General Conclusions

The preceding analysis has shown that gravitational effects can be separated from inertial disturbances in the second derivatives of the potential in case of inertial stabilization, and in the third derivatives in the absence of stabilization. This separation is possible because of the locally different behaviour of gravitational and inertial fields.

In the first derivatives, forming the force vector, such a separation by means of mechanical principles is a priori impossible because of the identity of inertial and gravitational mass. (It is, however, possible to remove inertial disturbances to a large extent by a suitable statistical filtering.)

Thus, with the advent of measuring techniques using equipment in motion, the higher derivatives of the gravitational potential may well become of increasing use to geodesists.[1] Therefore, we have discussed some relevant geodetic aspects in Chapter 1.

It should be carefully kept in mind that the gravitational potential cannot be determined without external information as to position, etc., even if the second derivatives can be obtained directly. The reason is that integrations such as (5) and (6) or (32) require the knowledge of the position of the measuring instrument, for instance of the form (4). This position cannot be determined by inertial navigation only because inertial navigation systems cannot discriminate between gravitational

[1] A measuring technique is discussed in (Thompson, Bock, and Savet, 1965).

and inertial accelerations, which are the input; we would thus be led to a vicious circle involving a bootstrap operation as mentioned in the Introduction.

In the theoretical analysis of Chapter 2 we have restricted ourselves to the simplest case of inertial stabilization, in which the instrument axes are kept parallel to axes fixed in inertial space. The results as to separability in the second derivatives hold, of course, also for the case in which the instrumental axes are stabilized in such a way as to have a prescribed orientation with respect to the earth (in this case, the matrix w_{ij} is different from zero but given, so that the right-hand side of (68) is known).

The relativistic analysis of the separability of gravitation and inertia has provided an important confirmation of the results obtained through classical mechanics. This analysis was necessary to remove the doubts raised by Einstein's Principle of Equivalence.

The general theory of relativity is commonly accepted as the best theory of gravitation, and it is therefore natural that geodesy, which uses gravitation to a large extent, should be interested in it. Furthermore, it has a particular appeal to geodesists because the geometric principle of Einstein's theory of gravitation is an extension of Gauss' surface theory, familiar to every geodesist, to four-dimensional space-time, and the relevant mathematical technique, tensor calculus, has been introduced into geodesy already some time ago by the work of Marussi and Hotine.

General relativity has so far been widely applied to problems of astronomy, and it similarly lends itself to application to kinematical geodesy; as an example, consider the "kinematic gravity potentials" (sec. 2.3.1) which generalize the gravity potential in a way necessary to take account of inertial forces acting on moving systems. Matters may be somewhat comparable to the use of Hamiltonian methods in satellite geodesy: although these methods are considerably more complicated mathematically than the simple Newtonian theory, they are much more efficient for the solution of some practical problems.

The foundation for an application of the general theory of relativity to geodetic problems may be found in (Synge, 1960); the tendency, the mathematical methods, and the applications are particularly appealing to astronomers and geodesists. As an example we mention Synge's "relativistically valid geodetic survey," which we have used in sec. 2.3.4.

Relativistic methods should be useful in problems such as the kinematical measurement of gravitational effects or the interaction of inertial positioning systems with the gravity field, with which geodesy is likely to be confronted in the near future.

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With the use of moving instruments, such as airborne gravimeters, which are often related to precise inertial stabilization systems, the essentially statical character of geodesy begins to be enlarged by kinematical features. The potential of gravity, combining gravitational attraction and centrifugal force, is no longer adequate for kinematics, but other inertial forces of Coriolis type must also be considered.

The main purpose of the present paper is an investigation of the geodetic aspects of the interrelation of gravitational and inertial forces and their separation by means of structural differences in their respective fields. For a deeper insight, the general theory of relativity is indispensable; it also furnishes convenient mathematical techniques.

The extraction of purely g[illegible]ational effects is possible with second and third derivatives of the potential; therefore geodetic applications of these quantities are discussed, integral formulas similar to Stokes' integral being given for this purpose.

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